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EXPONENTIALLY SMALL COUPLINGS BETWEEN TWISTED FIELDS OF ORBIFOLD STRING THEORIES

by

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Abstract:

We investigate the natural occurrence of exponentially small couplings in effective field theories deduced from higher dimensional models. We calculate the coupling between twisted fields of the Z_3 Abelian orbifold compactification of the heterotic string. Due to the propagation of massive Kaluza-Klein modes between the fixed points of the orbifold, the massless twisted fields located at these singular points become weakly coupled. The resulting small couplings have an exponential dependence on the mass of the intermediate states and the distance between the fixed points.

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I Introduction

Many phenomena explained by effective field theories require small couplings or small mass ratios. This is for instance the case of the hierarchy problem where there is a large discrepancy between the weak scale m_W and the Planck mass. A possible solution to this puzzle would be to introduce very small couplings in the Lagrangian of the standard model. However these small couplings are very difficult to motivate from the point of view of field theory. Indeed, most low energy effective field theories are expected to be natural, i.e all couplings which are not forbidden for symmetry reasons are of order $O(1)$[1]. One would therefore like to investigate the mechanisms which could help generating very small couplings. A possible scenario can be envisaged when the low energy effective field theories are approximations of more fundamental theories with extra dimensions[2]. The presence of these extra dimensions entails the existence of massive Kaluza-Klein modes upon compactification. These modes couple fields living far apart in the extra dimensions. These fields become very weakly coupled due to the small correlation length implied by the massive Kaluza-Klein modes. These small couplings appear when the compactifying space exhibits curvature singularities where fields are located. A natural realisation of this mechanism is provided by the compactification of the heterotic string on an orbifold[3]. Indeed the action of the point group of the orbifold on the six dimensional torus has fixed points leading to singularities . Each of these fixed points is associated to massless twisted states. Massive untwisted solitonic strings can propagate between the fixed points[4]. This entails the existence of exponentially small couplings between twisted states at different fixed points[5,6].

In section II, we consider the case of a free field theoretic model compactified on a singular D dimensional torus. The effective interaction between fields situated at two different singularities is exponentially small due to the exchange of one massive mode between the fixed points. In section III we extend our analysis to orbifold string models. Choosing the case of the Z_3 orbifold, we show that exponentially small couplings between twisted states are present. We have included an appendix where we derive some useful results on free propagators.

II Exponentially Small Couplings and Toroidal Compactification

Let us consider the interaction between a complex scalar field Ψ in $(d+D)$ dimensions and two complex scalar fields ϕ_1 and ϕ_2 assigned to two fixed points in the extra D dimensions, i.e. $\phi_{1,2}(x,y) = \phi_{1,2}(x)$ if $y = f_{1,2}$ and zero otherwise where y parametrises the extra dimensions. The two fixed points $f_{1,2}$ are singularities in the extra dimensions. We assume that the extra dimensions are flat apart from a curvature singularity at the

fixed points, i.e. $g^{\frac{1}{2}}R = \delta(x_{d+1} - f_1) + \delta(x_{d+1} - f_2)$. The action involves the kinetic terms

$$S_{kin} = \int d^dx \; |\partial_\mu \phi_1(x)|^2 + \int d^dx \; |\partial_\mu \phi_2(x)|^2 + \int d^{d+D}x \Psi^*(-\partial_A^2 - M^2)\Psi \qquad (1)$$

where $A = 1..(d+D)$ and the Minkowsky space-time index $\mu = 1..d$. The mass M is the mass of the scalar field Ψ. The interation term reads

$$S_{int} = -\lambda \Lambda^{\frac{(4+D-d)}{2}} \int d^{d+D}x \sqrt{g} R \Psi(\phi_1^2 + \phi_2^2) + hc \qquad (2)$$

where Λ is the scale where the model is defined and λ a positive dimensionless constant. We assume that the extra dimensions are curled up on a torus. This amounts to compactifying the model on a lattice $2\pi L$ where L is integral. This can serve as a prelude to the string calculation. The scattering amplitude $\phi_1 + \phi_1 \to \phi_2 + \phi_2$ is given by the propagator of the massive particle propagating between the two singularities $T = -\frac{\lambda^2 \Lambda^{4+d-d}}{2} G$. We shall derive several useful representations for the propagator

$$G =< f_2|(\frac{M^2 - s - \Delta_D}{2})^{-1}|f_1 > \qquad (3)$$

which will reappear in an appropriate guise in string theory.

First of all, let us use the completeness relation $\sum_{p \in L^*} |p >< p| = 1$ where the sum is over the dual lattice $L^* = \{p/ \; p.x \in Z, z \in L\}$. This yields

$$< f_2|(\frac{M^2 - s - \Delta_D^2}{2})^{-1}|f_1 >= V_L^{-1} \int_0^\infty dt_E e^{-\frac{(M^2 - s)}{2} t_E} \sum_{p \in L^*} e^{2\pi i p.(f_1 - f_2)} e^{-\frac{p^2}{2} t_E} \qquad (4)$$

where $|p >$ is an eigenstate of $-\Delta_D$ and

$$< f_2|e^{t_E \frac{\Delta}{2}}|f_1 >= V_L^{-1} \sum_{p \in L^*} e^{2\pi i p.(f_1 - f_2)} e^{-\frac{p^2}{2} t_E} \qquad (5)$$

. The propagator corresponds to the propagation of a Gaussian wave packet in Euclidean time from f_1 to f_2. Let us define the Hilbert space $\mathcal{H}$ spanned by the eigenstates $|p >$. One can reformulate the heat kernel as

$$< f_2|e^{t_E \frac{\Delta_D}{2}}|f_1 >= Tr_{\mathcal{H}}(e^{2\pi i P.(f_1 - f_2)} e^{-t_E H}) \qquad (6)$$

where $P = i\nabla$ is the momentum operator and $H = \frac{P^2}{2}$ is the Hamiltonian. This Hamiltonian version of the propagator can be written as a path integral over classical trajectories. Indeed, we can use the Poisson resummation formula to obtain a sum over the lattice L. Using

$$e^{-\frac{p^2 t_E}{2}} = \frac{1}{(2\pi t_E)^{\frac{D}{2}}} \int d^Dx e^{ip.x} e^{-\frac{x^2}{2t_E}} \qquad (7)$$

and the resummation formula

$$\sum_{p\in L^*} e^{ip.x} = V_L \sum_{q\in L} \delta(x-2\pi q) \tag{8}$$

one gets

$$G = \sum_{q\in L} \int_0^\infty dt_E \frac{1}{(2\pi t_E)^{\frac{D}{2}}} e^{-t_E \frac{(M^2-s)}{2}} e^{-(2\pi)^2 \frac{(f_1-f_2-q)}{2t_E}} \tag{9}$$

Finally, this is equivalent to the path integral

$$G = \sum_{q\in L} \int_0^\infty dt_E e^{-t_E \frac{(M^2-s)}{2}} \int_{2\pi f_1}^{2\pi f_2+2\pi q} \mathcal{D}x e^{-\frac{1}{2}\int_0^{t_E} \dot{x}^2 d\tau} \tag{10}$$

The sum is due to the periodicity of the lattice. If $s < M^2$ this integral converges. In the vicinity of $s = M^2$, there is a pole and the propagator is dominated by the $p = 0$ term in (4)

$$< f_2 | \frac{1}{M^2 - s - \Delta_D} | f_1 > \sim \frac{V_L^{-1}}{M^2 - s} \tag{11}$$

This comes from the fact that the propagator is solution of

$$(M^2 - s - \Delta_D) G = \sum_{q\in L} \delta(x - 2\pi q) \tag{12}$$

Using the resummation formula (8) and writing $G(x) = \sum_{p\in L^*} G_p e^{ip.x}$ one gets

$$\sum_{p\in L^*} (m^2 - s + p^2) G_p e^{ip.x} = V_L^{-1} \sum_{p\in L^*} e^{ip.x} \tag{13}$$

It is easy to see that $G_0 = \frac{V_L^{-1}}{M^2-s}$. Notice that the pole disappears as $V_L \to \infty$, it is a finite size effect. Below M^2, the propagator receives contributions from the whole series. In fact, one can use a saddle point approximation in this regime (more details are given in the appendix) As a result, the propagator reads

$$< f_2 | \frac{1}{m^2 - s - \Delta_D} | f_1 > \sim \frac{1}{2^D (M^2 - s)^{\frac{D}{2}}} \sum_{q\in L} e^{-2\pi\sqrt{M^2-s}|f_2 - f_1 + q|} \tag{14}$$

below the pole. This result specialised for $s \ll M^2$ gives the value of the coupling between the two fixed points

$$V_{coup} = -\frac{\lambda^2 \Lambda^{4+D-d}}{(2M)^D} \sum_{q\in L} e^{-2\pi M |f_2 - f_1 + q|} (\phi_1^2 \phi_2^{*2} + c.c.) \tag{15}$$

One can immediately generalise this result to the gauge model where the field $\phi_{1,2}$ transform in the vectorial representation of a gauge group G

$$V_{coup} = -\frac{\lambda^2 \Lambda^{4+D-d}}{(2M)^D} \sum_{q \in L} e^{-2\pi M |f_2 - f_1 + q|} (\bar{\phi}_1 T^a \phi_1)(\bar{\phi}_2 T^a \phi_2) \tag{16}$$

where T^a are the generators of the gauge group. We see that the compactification on a singular torus leads to exponentially small couplings in the effective theory describing the uncompactified dimensions.

II Small Couplings in Orbifold String Theories

Orbifold string theories provide an elegant and tractable scheme where stringy calculations can be performed[7]. Moreover, there are singularities on an orbifold which are of the type discussed in the previous section, i.e. four dimensional twisted states are assigned to the singularities of the orbifold. It seems natural to look for small couplings between states living far apart at different fixed points. The analogue of the massive field Ψ is the tower of untwisted states which can propagate between fixed points.

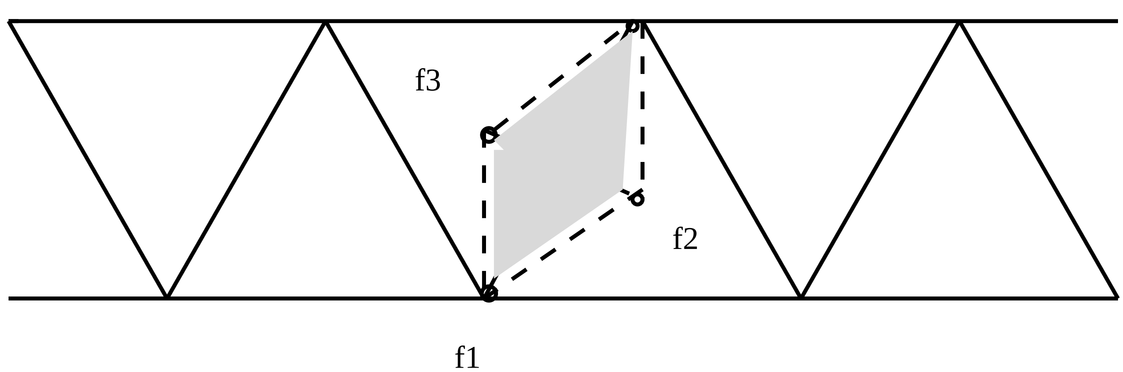

Fig 1: The Z3 lattice, the three fixed points f1, f2 and f3. The shaded area is the Z3 orbifold obtained after folding the lattice identifying points under the Z3 symmetry.

We shall consider the $E_8 \times E_8$ heterotic string theory compactified on an Abelian orbifold. The left part of the theory contains the gauge degrees of freedom while the right part is supersymmetric on the world sheet and is responsible for the spin content of the spectrum. The compactification from 10 to 4 dimensions is obtained by considering the theory on a six dimensional torus on which acts an Abelian group of automorphisms of the corresponding lattice L. As the automorphism group does not act freely on the torus, the resulting quotient space is not a manifold. There are singularities associated to the fixed points of the group of automorphism. When analysing the massless spectrum of the theory, one finds that two types of states are present. The untwisted states correspond the dimensional reduction of 10 dimensional states whereas the twisted states correspond to closed strings satisfying

$$X_i(\sigma + 2\pi) = \theta^j X_(\sigma) + \Lambda_i \tag{17}$$

where X_i are the six bosonic coordinates on the orbifold, θ is the generator of the abelian group in the Z_N case and Λ is an element of the lattice L. These states are in the jth twisted sector. Each of these states is associated to one of the fixed points of the orbifold. This implies that the center of mass of these states is at one of the fixed points f. Twisted states give a natural realisation of the fields $\phi_{1,2}$ of the previous analysis.

The gauge group of the $(N = 1)$ model is determined by the left part of the orbifold string theory. The left part contains 16 scalar fields or equivalently 16 complex fermions which can be bosonised on the $E_8 \times E_8$ lattice $\lambda_i \sim e^{i\beta_I^i F_I}$ where $i = 1..16$, $I = 1..16$ and $\beta = (\underline{\pm 1, 0, 0...., 0, 0})$. The gauge bosons are determined by $e^{iP.F}$ where P is a root of the gauge lattice, i.e. $P^2 = 2$. The generator of the abelian group automorphisms θ acts simultaneously on the gauge lattice by translation. The left part of the gauge boson vertex operator is invariant under the action of θ. This implies that $P.V \in Z$. The choice of the gauge group can be even more constrained by considering Wilson lines[8] A_i^I, $I = 1..16, i = 1..6$. Integrating the Wilson lines along one of the six cycles of the torus defined by L gives $\int_i dx_m A_I^m = 2\pi a_I^i$. The gauge group is restricted as the root vectors must satisfy $P.a^i \in Z$. Wilson lines also modify the twisted spectrum . For the Abelian Z_3 orbifold, the lattice vectors determining the spectrum satisfy $(P + nV + ma)^2 = \frac{4}{3}$ where $n = 0..2$ labels the twisted sector and $m = 0..2$ distinguishes the different fixed points. We shall use the Z_3 orbifold and its 27 fixed points.

We are interested in exponentially small terms in the scattering of four twisted states. Indeed the scattering amplitude between these states is the stringy analogue of the previous field theoretic calculations where fields lived at singular points in the compactified dimensions. Fixed points being separated by a distance proportional to the orbifold radius, one can expect that the scattering amplitude is exponentially suppressed due to the exchange of massive string modes. The calculation of the scattering amplitude involving four twisted states gives information on renormalisable interaction terms in four dimensions. Higher order scattering amplitude would lead to non-renormalisable interaction terms. The calculation will be done at tree level on the sphere. We closely follow the original analysis of Dixon et al. in ref.[5].

Let us explicitly calculate the scattering amplitude in the case of a standard embedding of the space group in the gauge lattice $V = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0, 0, 0, 0, 0)$. One can choose one twisted field in the 27 of E_6 and one anti-twisted field in the $\bar{27}$ living at the fixed point

f_1, while two other 27 and $\bar{27}$ are at another fixed point f_2. The correlation function $< 27(f_2)\bar{27}(f_2)27(f_1)\bar{27}(f_1) >$ is not forbidden by the point group selection rule (two twisted and two anti-twisted fields are θ invariant) and the space group selection rule (the combination of fixed points satisfies $f_1 + f_2 - f_1 - f_2 = 0$). Using conformal invariance, one can send three points to $0, 1$ and ∞. The fourth argument of the correlation function is the cross ratio x. Let us compute

$$Z(x) = \lim_{z_\infty \to \infty} |z_\infty|^4 < \bar{27}_{f_2}(z_\infty) 27_{f_2}(1) 27_{f_1}(0) \bar{27}_{f_1}(x) > \tag{18}$$

where $27_{f_1}(0)$ is the vertex operator of the field in the 27 of E_6 located at the origin of the sphere and embedded in the orbifold at the fixed point f_1. Vertex operators are a product of a left part which contains the gauge degrees of freedom with a right part involving the $so(10)$ supersymmetric content of the fields. The vertex operators of a 27 twisted field looks like

$$27_{L,f_1}(0) = e^{i(P+V).F} \Lambda_{+,f_1}(0) \tag{19}$$

(respectively $-P - V$ for an anti-twisted $\bar{27}$). The twist operator is denoted by $\Lambda_{+,f_1}(0)$. The massless states satisfy $(P + V)^2 = \frac{4}{3}$, i.e. P can be equal to

$$\begin{array}{c} (0, 0, 1, \underline{\pm 1, 0, 0, 0, 0}) \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \underline{\pm\frac{1}{2}, 0, 0, 0, 0}) \\ (-1, -1, 0, 0, 0, 0, 0, 0) \end{array} \tag{20}$$

The first possibility builds up the 10 of $so(10)$ while the second gives the 16 and the last is a singlet. The left part of the correlation function (18) reads

$$< e^{i(P_1+V).F(0)} e^{-i(P_2+V).F(\bar{x})} e^{-i(P_3+V).F(\bar{z}_\infty)} e^{i(P_4+V).F(1)} > Z_{twist}(\bar{z}_\infty) \tag{21}$$

where

$$Z_{twist}(\bar{z}_\infty) =< \Lambda_{+,f_1}(0) \Lambda_{-,f_1}(\bar{x}) \Lambda_{-,f_2}(\bar{z}_\infty) \Lambda_{+,f_2}(1) > \tag{22}$$

The correlation function is non-zero if $P_1 + P_4 = P_2 + P_3$. An easy calculation gives

$$\bar{z}_\infty^a \bar{x}^b (1 - \bar{x})^c Z_{twist}(\bar{z}_\infty) \tag{23}$$

where

$$\begin{aligned} a &= -(P_3 + V)^2 \\ b &= -(P_2 + V).(P_1 + V) \\ c &= -(P_2 + V).(P_4 + V) \end{aligned} \tag{24}$$

The exponents depend on the choice of combinations of P's. We shall choose them in such a way that $P_1 - P_2$ correspond to the adjoint representation in the product

$$\underline{27} \times \overline{27} = \underline{1} + \underline{78} + \underline{650} \tag{25}$$

in the s channel. The adjoint representation $\underline{78}$ is defined by the root lattice of E_6 such that $(P_1 - P_2)^2 = 2$, for instance $P_1 = (0,0,1,1,0,0,0,0,0)$ and $P_2 = (0,0,1,0,1,0,0,0)$. The exponents become

$$a = -\frac{4}{3}$$
$$b = -\frac{1}{3} \tag{26}$$
$$c = -\frac{1}{3}$$

These exponents are determined once two constraints are imposed. First of all the limit $x \to 0$ must show a pole in the s channel corresponding to the exchange of a gauge boson. Then the limit $x \to \infty$ must correspond to a non-vanishing Yukawa coupling between three twisted states. For other embeddings of the space group of the orbifold in the gauge lattice and if Wilson lines are included, the twisted states belong to various representations of a gauge group G. If one assumes that there exists a gauge invariant combination of three twisted states involving f_1, f_2 and another fixed point whose coupling is not forbidden by the space group rule, then the left part of the correlation function is still given by (23).

Let us now deal with the right part of the correlation function. The vertex operators depend on the picture chosen. In the -1 picture the vertex operators read

$$27_{+,f_1,R,-1}(0) = e^{-\phi(0)} e^{i(\alpha_v + v).H(0)} \Lambda_{+,f_1}(0) \tag{27}$$

where $\alpha_v = (0,0,1,0,0)$. In the 0 picture, the part of the vertex operator yielding a non-zero 4-point function depends on the four-momentum of the state

$$\bar{27}_{-,f_1,R,0}(x) = ik_2.\psi(x) e^{-i(\alpha_v+v).H(x)} \Lambda_{-,f_1}(x) \tag{28}$$

We can assign the states at the origin and 1 to be in the -1 picture whereas states at x and ∞ are in the 0 picture (the sum of the charges is -2). Then using $< \psi_\mu(z_\infty)\psi_\nu(x) > \sim \frac{\eta_{\mu\nu}}{z_\infty}$ and $< e^{-\phi(1)} e^{-\phi(0)} >= 1$ as $\dim(e^{-\phi}) = \frac{1}{2}$, one gets

$$k_2.k_3 z_\infty^{-\frac{4}{3}} x^{-\frac{1}{3}} (1-x)^{-\frac{1}{3}} Z_{twist}(z_\infty) \tag{29}$$

where we have used $(\alpha_v + v)^2 = \frac{1}{3}$ and denoted by $Z_{twist}(z_\infty)$ the right part of the twist correlation function. The correlation function reads

$$Z(x) = k_2.k_3 |1-x|^{-\frac{2}{3}-u} |x|^{-\frac{2}{3}-s} Z_{twist} \tag{30}$$

where

$$Z_{twist} = \lim_{z_\infty \to \infty} |z_\infty|^{\frac{4}{3}} < \Lambda_{+,f_1}(0)\Lambda_{-,f_1}(x,\bar{x})\Lambda_{+,f_2}(z_\infty,\bar{z}_\infty)\Lambda_{+,f_2}(1) > \tag{31}$$

and $s = (k_1 + k_2)^2$, $u = (k_1 - k_4)^2$ are the Mandelstam variables.

The twist correlation function is obtained by calculating the path integral of bosonic strings on the orbifold satisfying the boundary conditions prescribed by the four twisted states at $0, 1, x$ and ∞. One can separate the bosonic string into a quantum and a classical

part. The classical strings represent untwisted states propagating between the fixed points. The quantum part corresponds to the fluctuations around the classical strings, in particular the quantum part of the correlation function reproduces the leading exponents in the operator product expansion as $x \to 0, 1$, i.e $Z(x) = Z_{qu} Z_{cl}$ where

$$Z_{qu} = \sin \frac{\pi}{3} V_L \frac{|x|^{-\frac{4}{3}} |1-x|^{-\frac{4}{3}}}{I^3} \tag{32}$$

More precisely, the quantum part depends on the hypergeometric function

$$F(x) = \frac{\sin \frac{\pi}{3}}{\pi} \int_0^1 dy \frac{1}{y^{\frac{1}{3}} (1-y)^{\frac{2}{3}} (1-xy)^{\frac{1}{3}}} \tag{33}$$

as $I = F(x)\bar{F}(1-\bar{x}) + \bar{F}(\bar{x})F(1-x)$. One also needs to define $\tau = i\frac{F(x)}{F(1-x)}$. This gives

$$Z(x) = \sin \frac{\pi}{3} V_L k_2.k_3 \frac{|1-x|^{-2-u} |x|^{-2-s}}{I^3} Z_{cl} \tag{34}$$

We are therefore left with the evaluation of the classical action of untwisted string on the orbifold

$$Z_{cl} = \sum e^{-S} \tag{35}$$

where the sum is over the set of classical solutions with classical action S. Let us now determine this classical solutions.

Recall that the bosonic string action in the conformal gauge reads

$$S = \frac{1}{4\pi} \int d^2 z (\partial X \bar{\partial} \bar{X} + \bar{\partial} X \partial \bar{X}) \tag{36}$$

where $X = (X^1, X^2, X^3)$ is a complex scalar field build from the six bosonic fields $X_i, i = 1..6$ of the orbifold, e.g. $X^1 = X_1 + iX_2$. The variable z lives on the Riemann sphere. The path integral yielding the scattering amplitude is dominated by classical configurations characterised by the Laplace equation

$$\Delta X = 0 \tag{37}$$

These instantons are constrained by appropriate boundary conditions. They are classified according to their degree. Indeed, they represent maps from the Riemann sphere to the orbifold. As the orbifold Z_3 is homeomorphic to a 2-sphere, one therefore deduces that the instantons are maps belonging to $\Pi_2(S^2) = Z$. Each of these instantons is specified by one integer, i.e. its degree. The degree describes the number of time the closed strings wraps around the orbifold. We shall see that instantons of any degree exist.

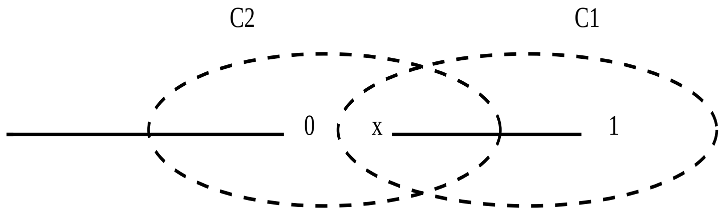

Fig 2: The contours C1 and C2 and the cuts defining the 2-torus

As the Riemann sphere has four marked points where the twisted states are attached $0, 1, \infty$ and x, the instantons are multivalued functions on the sphere. One can construct a 3-sheeted covering of the sphere with four branch points of order 3 where the instanton is single-valued (locally, the twist field introduce a cubic root close to each branch point). The corresponding Riemann surface has a genus given by the Riemann-Hurwitz formula $2g-2 = 3*(-2)+4*2$, so the genus is $g=2$. The solutions of the Laplace equation are obtained once a basis for the Abelian differentials are known on the 2-torus. The dimension of the space of Abelian differentials is 2 spanned by

$$\begin{aligned} \omega_1 &= \frac{dy}{y^{\frac{1}{3}}(1-y)^{\frac{1}{3}}(x-y)^{\frac{2}{3}}} \\ \omega_2 &= \frac{dy}{y^{\frac{2}{3}}(1-y)^{\frac{2}{3}}(x-y)^{\frac{1}{3}}} \end{aligned} \tag{38}$$

The 2-torus is obtained by considering the complex plane with two cuts, one between $-\infty$ and 0 along the real axis, the other one between x and 1 along the real axis. There are four independent loops on the two-torus. In fact they can be all generated from two loops in the complex plane $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively encircling the cut from x to 1 and going across the cuts to encircle the segment between 0 and x. Four other loops can be obtained by transferring them to the two other sheets. One can select four of these loops to form a homology basis of the 2-torus.

We can now discuss the boundary conditions for the untwisted strings propagating between the fixed points. These boundary conditions completely specify the instantons. The instantons are determined by their monodromy properties around the closed loops of the two-torus with net twist zero (they are untwisted). Indeed, consider the two twisted strings at x and the origin, they satisfy

$$\begin{aligned} X_0(\sigma+2\pi) &= \theta X_0(\sigma) + (1-\theta)(f_1+q_0) \\ X_x(\sigma+2\pi) &= \theta^{-1} X_x(\sigma) + (1-\theta^{-1})(f_1+q_x) \end{aligned} \tag{39}$$

when transported along a closed loop surrounding 0 (respectively x). When these two twisted string merge to create a single untwisted closed string, the monodromy of the

resulting closed string X when transported around a closed loop surrounding x and 0 is obtained by combining $X(\sigma+2\pi) = \theta(\theta^{-1}X(\sigma)+(1-\theta^{-1})(f_1+q_x))+(1-\theta)(f_1+q_0)$. The resulting string is untwisted with a winding number $v_2 = (1-\theta)(q_0-q_x)$ (this phenomenon is sketched on fig. (4)). Similarly, one has to deal with boundary conditions involving the other external strings at 1 and ∞. As the homology of the 2-torus is spanned by two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$, one only needs to study the monodromy of the untwisted string wrapped along these two contours. The contour $\mathcal{C}_2$ corresponds to the previous case. One can deal with the other case to obtain

$$\Delta_{\mathcal{C}_i} X = \int_{\mathcal{C}_i} \partial X + \int_{\mathcal{C}_i} \bar{\partial} X = 2\pi v_i \tag{40}$$

where $\mathcal{C}_i$ are the independent loops of the 2-torus. The vectors v_i belong to certain cosets of the lattice L of the orbifold, $v_1 \in (1-\theta)(f_2 - f_1 + L),\ v_2 \in (1-\theta)L$.

The instantons are solutions of

$$\begin{aligned} \partial X &= a\omega_1 \\ \bar{\partial} X &= b\bar{\omega}_2 \end{aligned} \tag{41}$$

Notice that ω_1 and $\bar{\omega}_2$ pick up a $\theta = e^{i\frac{2\pi}{3}}$ phase factor when crossing the cuts. We only need to check the monodromies for the loops $\mathcal{C}_1$ and $\mathcal{C}_2$ as on the other sheets $\Delta_{\theta\mathcal{C}_i} X = 2\pi\theta v_i$ is automatically satified as θv_i belong to the same coset as v_i. In order to simplify the integrals, we shall assume that x is real positive. The general case is easily dealt with. The coefficient a and b are determined using the periods

$$\begin{aligned} \int_{\mathcal{C}_1} \omega_1 &= -2\pi i F(1-x) \\ \int_{\mathcal{C}_1} \bar{\omega}_2 &= 2\pi i F(1-x) \\ \int_{\mathcal{C}_2} \omega_1 &= -2\pi i \theta^{\frac{1}{2}} F(x) \\ \int_{\mathcal{C}_2} \bar{\omega}_2 &= -2\pi i \theta^{\frac{1}{2}} F(x) \end{aligned} \tag{42}$$

Writing $a = a_1 v_1 + a_2 v_2$ and $b = b_1 v_1 + b_2 v_2$ one gets

$$\begin{aligned} a_1 = -b_1 &= \frac{i}{2F(1-x)} \\ a_2 = b_2 &= \frac{i\theta^{-\frac{1}{2}}}{2F(x)} \end{aligned} \tag{43}$$

The instanton is obtained by integrating (41). Let us first integrate (41) starting from a real point on the right of 1. One draws an infinitesimal half-circle around each branch point.

Let us analyse the images of the branch points $1, x, 0$ and ∞. In general the instanton is given by

$$X(z) = 2\pi f_2 + a_1 v_1 (\int_1^z \omega_1 - \int_1^z \bar{\omega}_2) + a_2 v_2 (\int_1^z \omega_1 + \int_1^z \bar{\omega}_2) \tag{44}$$

Now an easy calculation gives

$$X(x) = 2\pi f_2 - \frac{2\pi i}{\sqrt{3}} \theta^{\frac{1}{2}} v_1 \tag{45}$$

This point is identified with the fixed point $2\pi f_1$. Similarly, the image of 0 is

$$X(0) = 2\pi f_1 + \frac{2\pi i \theta^{-\frac{1}{2}}}{\sqrt{3}} v_2 \tag{46}$$

This is the same fixed point as one has added a vector of the lattice L. Finally the image of ∞ is simply

$$X(\infty) = 2\pi f_1 + \frac{2\pi i}{\sqrt{3}} \theta^{\frac{1}{2}} v_1 \tag{47}$$

which is nothing but $2\pi f_2$. These four points are the four vertices of a parallelogram. The image of the instanton on the orbifold is then easily deduced. Notice first that the instanton is a multivalued function on the complex plane. As one goes around $\mathcal{C}_{1,2}$, the image of the same point on the sphere is shifted by $2\pi v_{1,2}$. One can focus on the image of the sphere modulo the lattice generated by $2\pi v_1$ and $2\pi v_2$. The image is in a bounded domain. Moreover the image is repeated three times as one goes from one sheet to another. Each sheet is sent to one of the images of the sphere. The images of the other two sheets is obtained by rotating the image of one sheet by θ and θ^2. Hence we see that the image of the instanton is obtained by restricting () to one sheet modulo the period lattice generated by $2\pi v_{1,2}$. Once the images of the four marked points is known, the image of the instanton is the unfolding of the orbifold passing through the marked points. Indeed the image of the instanton is a closed surface, i.e. a multiple covering of the orbifold. The degree of the instanton is given by the number of copies of the orbifold necessary to join the four marked points. One also has to make sure that the complete image of the instanton obtained by translating the image of one sheet by $2\pi(nv_1+mv_2)$ form a connected set. Several examples are given in Fig.3.

We can now calculate the classical action using $\int d^2z|\omega_{1,2}|^2 = \frac{2\pi^2}{\sin\frac{\pi}{3}} F(x)F(1-x)$. This yields

$$S = \frac{\pi}{4\mathrm{Im}\tau \sin\frac{\pi}{3}} (|v_2|^2 + |\tau|^2|v_1|^2) \tag{48}$$

When x is complex one gets

$$S = \frac{\pi}{4\mathrm{Im}\tau \sin\frac{\pi}{3}} (|v_2|^2 + \mathrm{Re}\tau(v_1\bar{v}_2\bar{\beta} + v_2\bar{v}_1\beta) + |\tau|^2|v_1|^2) \tag{49}$$

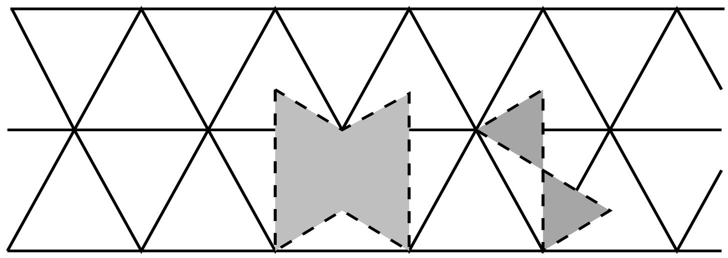

Fig 3: Two unfolded instantons on the Z3 orbifold

The shaded areas represent instantons of degree one

and degree three.

where $\beta = -\theta^{\frac{1}{2}}$. The classical part of the correlation function is obtained by summing over all the boundary conditions specified by v_1 and v_2.

$$Z_{cl} = \sum_{v_1, v_2} e^{-\frac{\pi}{4Im\tau \sin\frac{\pi}{3}}(|v_2|^2 + Re\tau(v_1\bar{v}_2\bar{\beta} + v_2\bar{v}_1\beta) + |\tau|^2|v_1|^2)} \tag{50}$$

The scattering amplitude is given by

$$T = \int d^2x Z(x) \tag{51}$$

We devote the next section to the computation of the scattering amplitude in the s-channel when $x \to 0$. One expects this channel to be sensitive to the exchange of a gauge boson. We shall see that the s channel bears a perfect analogy with the field theoretic calculations of the previous section. It is interesting to reformulate the scattering amplitude in terms of operators and states in a Hilbert space. This will allow us to determine the intermediate states dominating the scattering amplitude. In the limit $x \to 0$ using $F(x) \sim 1$ and $F(1-x) \sim -\frac{\sin\frac{\pi}{3}}{\pi} \ln x$ and the Poisson resummation formula, one can obtain $Z(x)$ as a sum over momenta in the dual lattice $p \in L^*$ and winding vectors $v_2 \in L$

$$Z(x) \sim -u|x|^{-2-s} \sum_{v_2 \in (1-\theta)L,\ p \in L^*} e^{2\pi i(f_1 - f_2).p} x^{\frac{(p+\frac{v_2}{2})^2}{2}} \bar{x}^{\frac{(p-\frac{v_2}{2})^2}{2}} \tag{52}$$

One can compare this expansion with the result obtained using the operator product expansion of $\bar{27}(x,\bar{x})27(0)$ and identify

$$\begin{aligned} \bar{27}(x,\bar{x})27(0)|0> = & \sum_{p \in L^*,\ v_2 \in (1-\theta)L} e^{2\pi i(f_1-f_2).p} x^{H_R - \frac{s}{2} - 1} \bar{x}^{\bar{H}_L - \frac{s}{2} - 1} \\ & ik_2.\psi(0) e^{-\phi(0)} e^{i(P_1-P_2).F(0)} e^{i(k_1+k_2).X(0)} \phi_{v_2,p}(0)|0> \end{aligned} \tag{53}$$

The operators $\phi_{v_2,p}(0)$ create states $|v_2, p>$ satisfying

$$\begin{aligned} H_L|v_2, p> &= \frac{1}{2}(p - \frac{v_2}{2})^2 |v_2, p> \\ H_R|v_2, p> &= \frac{1}{2}(p + \frac{v_2}{2})^2 |v_2, p> \end{aligned} \tag{54}$$

when $H_{L,R}$ are the left and right bosonic Hamiltonians on the orbifold. It is clear that the operators $\phi_{v_2,p}(z,\bar{z})$ are equal to

$$\begin{aligned} \phi_{v_2,p}(z) &= e^{i(p+\frac{v_2}{2}).X(z)} \\ \phi_{v_2,p}(\bar{z}) &= e^{i(p-\frac{v_2}{2}).\bar{X}(\bar{z})} \end{aligned} \tag{55}$$

where $X_i(z)$ is the six-dimensional field on the orbifold. The operators $\phi_{v_2,p}$ create winding states on the six dimensional torus with momentum p and winding number $2\pi v_2$. Indeed, let us define the momentum operator as

$$P_i = \frac{1}{4\pi}\int(\partial X_i dz - \bar{\partial}\bar{X}_i d\bar{z}) \tag{56}$$

This operator gives the momentum of strings propagating on the orbifold. Similarly, the winding number operator is

$$W_i = \int(\partial X_i dz + \bar{\partial}\bar{X} d\bar{z}) \tag{57}$$

It is easy to check that

$$\begin{aligned} P|v_2,p> &= p|v_2,p> \\ W|v_2,p> &= 2\pi v_2|v_2,p> \end{aligned} \tag{58}$$

Untwisted solitonic states are created in the s channel.

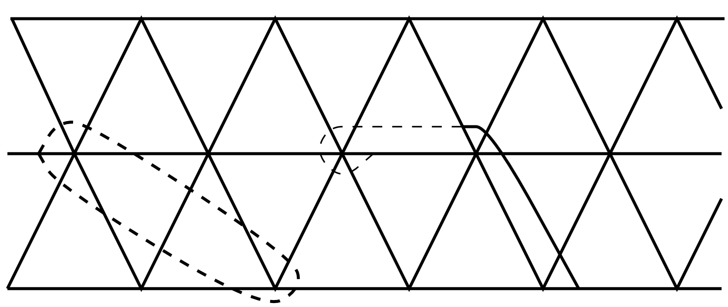

Fig. 4: A twisted string joining an antitwisted string to form an untwisted string. The twisted string is represented as a dashed line while the untwisted one is depicted as the thin full line. The resulting untwisted string has been shifted to the left using the lattice periodicity.

The scattering amplitude is obtained by integrating $Z(x)$ over x. Writing $x = |x|e^{i\sigma}$ one gets the integral

$$T \sim -u \int_{x\sim 0} d|x|d\sigma \sum_{p\in L^*,\ v_2\in(1-\theta)L} e^{2\pi ip.(f_1-f_2)} e^{i\sigma p.v_2} |x|^{p^2+\frac{|v_2|^2}{4}-1-s} \tag{59}$$

where we have used $-u-t=s$. The σ integral gives a Kronecker delta function $\delta(p.v_2)$. Putting $|x| = e^{-t_E}$, the scattering amplitude becomes

$$T \sim -u \int^{\infty} dt_E \sum_{p\in L^*, v_2\in(1-\theta)L} \delta(p.v_2) e^{2\pi ip.(f_2-f_1)} e^{-t_E(p^2+\frac{|v_2|^2}{4}-s)} \tag{60}$$

This formulation of the scattering amplitude is readily interpreted if one introduces the family of Hilbert spaces $\mathcal{H}_{v_2}$ spanned by the states $|v_2, p, \psi_\mu>$ appearing on the right hand side of the operator product expansion (53) for each v_2. It is easily to see that

$$(L_0 + \bar{L}_0 - 2)|v_2, p, \psi_\mu >= (p^2 + \frac{|v_2|^2}{4} - s)|v_2, p, \psi_\mu > \tag{61}$$

as well as

$$(L_0 - \bar{L}_0)|v_2, p, \psi_\mu >= (p.v_2)|v_2, p, \psi_\mu > \tag{62}$$

The scattering amplitudes therefore reads

$$T \sim -u \int^{\infty} dt_E \sum_{v_2\in(1-\theta)L} Tr_{\mathcal{H}_{v_2}}(\Gamma_{L_0-\bar{L}_0} e^{2\pi iP.(f_1-f_2)} e^{-t_E(L_0+\bar{L}_0-2)}) \tag{63}$$

where $\Gamma_{L_0-\bar{L}_0}$ is the projector on states satisfying $L_0 = \bar{L}_0$. We immediately recognise the propagator of strings in the Hilbert $\mathcal{H}_{v_2}$ with momentum operator P and Hamiltonian $L_0+\bar{L}_0-2$. The propagation is in Euclidean time. Moreover the propagation corresponds to the motion of 10-dimensional string from f_1 to f_2. One can match this Hamiltonian formulation of the scattering amplitude with the path integral approach. As for point particles, the propagation of a wave packet of states $|v_2, p, \psi_\mu>$ between the fixed points in Euclidean time is equivalent to the emission of a classical instanton characterised by its winding $2\pi v_2$ and surrounded by Gaussian fluctuations.

The scattering amplitude is similar to the one obtained in the field theoretical case. The only difference is the constraint $p.v_2 = 0$ corresponding to the translation operator along a closed string $U = e^{i\sigma(L_0-\bar{L}_0)}$. We can now directly apply our analysis of a free particle compactified on a torus. Let us rewrite the scattering amplitude by distinguishing the role of $p = 0$

$$\begin{aligned} T \sim & \sum_{v_2\in(1-\theta)L} \frac{u}{\frac{s-|v_2|^2}{4}} \\ & - u \int^{\infty} dt_E \sum_{p\in L^*-\{0\}, v_2\in(1-\theta)L} \delta(p.v_2) e^{2\pi ip.(f_2-f-1)} e^{-t_E(p^2+\frac{|v_2|^2}{4}-s)} \end{aligned} \tag{64}$$

As for free particles, one can see that in the vicinity of $\frac{|v_2|^2}{4}$ the scattering amplitude is dominated by poles corresponding to the emission of winding states of masses $\frac{|v_2|^2}{4}$. The first pole at $s=0$ is due to the exchange of a gauge boson of the gauge group G defined by $p=0,\ v_2=0$[9]. The other poles are all due to winding massive states. The sign of the pole is consistent with the result $T\sim(-1)^{S+1}\frac{u}{s-m^2}$ for the exchange of a particle of spin S and mass m. Here one knows that $S=1$ for vector bosons. Notice that these poles are not suppressed by the volume of the orbifold. Indeed, the normalisation is such that the effect of the gauge boson is still present is the large radius limit.

Away from these poles, we know that for each winding vector v_2 the behaviour of the partial scattering amplitude $T(v_2)$ depends on the energy s. Below the pole, the scattering amplitude is dominated by a saddle point corresponding to an instanton of winding vector v_2 while above the pole the saddle point point is due to a soliton. The only difference with the free particle case comes from the GSO projection. Using the poisson resummation formula, we get

$$T\sim -uV_L\int^{\infty}dt_E\int_0^{2\pi}d\sigma\frac{1}{t_E^3}\sum_{q_1\in L,v_2\in(1-\theta)L'}e^{-t_E(\frac{|v_2|^2}{4}-s)}e^{-\frac{2\pi(f_1-f_2-q_1+\sigma v_2)^2}{4t_E}} \tag{65}$$

where we have subtracted the contribution of the gauge boson for $v_2=0$ as $L'=L-\{0\}$. Let us concentrate on one particular v_2. Below the pole, the corresponding winding mode contributes as

$$T(v_2)\sim -uV_L\int_0^{2\pi}d\sigma\frac{1}{(\frac{|v_2|^2}{4}-s)^3}e^{-2\pi\sqrt{\frac{|v_2|^2}{4}-s}|f_1-f_2-q_1+\sigma v_2|} \tag{66}$$

where we have used a saddle point approximation as explained the appendix. Notice that the scattering amplitude is exponentially suppressed. This result is strikingly similar to the field theoretical result. The only difference comes from the average over the closed string parameter σ. It is interesting to remark that the saddle point equation reads

$$s+P^2=\frac{|v_2|^2}{4} \tag{67}$$

where P is the six-momentum of the instanton. The mass of each of these instantons is nothing but the mass of the corresponding quantum state propagating between the fixed points, i.e. $\frac{|v_2|}{2}$.

Let us come back on the resulting coupling in the effective field theory. We are interested in reproducing the above scattering amplitude in an effective theory. This effective theory depends on the scale at which one identifies the string scattering amplitude on the sphere with a coupling in the effective Lagrangian. The string calculation is performed at a scale $s\sim M_c^2$ close to the compactification scale. Dimensionally one has $M_c\sim\frac{M_{pl}}{R}$ where R is the radius of the orbifold in Planck units. Similarly the masses of the Kaluza-Klein modes is proportional to $|v_2|\sim RM_{pl}$. One can therefore neglect s compared to $|v_2|^2$ for

reasonable values of $R \geq 1$. Fixing the value of $s = M_c$ we find that there is a coupling in the effective Lagrangian of the form

$$\Delta V = \frac{T}{M_{pl}^2}(D_\mu \bar{\phi}_1 T^a \phi_1)(\bar{\phi}_2 T^a D^\mu \phi_2) \tag{68}$$

where D is the covariant derivative. The coupling contant T behaves like

$$T \sim V_L \sum_{v_2 \in (1-\theta)L - \{0\}} \int_0^{2\pi} d\sigma \frac{1}{|v_2|^6} e^{-\pi |v_2||f_1 - f_2 - q_1 + \sigma v_2|} \tag{69}$$

This new coupling is exponentially suppressed. The argument of the exponential is simply the mass of the Kaluza-Klein modes multiplied by the distance between the fixed points. This distance depends on the number of times the solitonic states wrap around the orbifold. Dimensionally, the exponent varies as R^2 where R is the radius of the orbifold. We have obtained contributions to the potential which are similar to the ones of the field theoretic compactification on a singular torus. Due to supersymmetry this exponentially small term modifies the Kahler potential, i. e. the kinetic terms. Notice that the coupling appears with two derivatives and is therefore suppressed by the Plank mass. The main difference between the field theoretic and stringy calculations stems from the nature of the intermediate states. In the field theoretic example, the exchanged particle is a Lorentz scalar whereas in the stringy case it is a Lorentz vector. This entails that the string coupling is derivative. It would be conspicuous to know if higher order scattering amplitudes yield non-derivative couplings occurring in the scalar potential.

VI Conclusion

We have been interested in naturally small couplings between massless states in effective field theories. We have shown that exponentially small couplings appear when massless fields located at singular points of a compactified space are coupled to massive states. These massive states propagate between the singular points leading to couplings depending exponentially on the masses of the intermediate states and the distance between the singularities. An interesting framework for this type of mechanism is provided by the heterotic string theory compactified on an orbifold. In that case the twisted states are located at the fixed points of the orbifold. These fixed points are singular. There exists an infinite tower of untwisted massive solitonic states coupling the fixed points. These Kaluza-Klein modes give rise to exponentially small couplings in the effective field theory. The existence of exponentially small couplings may be relevant to the hierarchy problem. Indeed one can hope that the large difference between the Planck scale and the weak scale is due to some small coupling whose origin would be a sign of extra stringy dimensions.

Appendix

In this appendix we shall compute the scattering amplitude $\phi_1+\phi_1\rightarrow\phi_2+\phi_2$ in the case $D=1$. In particular, we give a path integral formulation which reappears in the stringy calculation. One needs to evaluate the Green function of the operator $\partial_A^2+m^2$. Suppose that the d dimensional momentum of the particle is fixed $s=p^2$. One is therefore left with the operator $m^2-s-\partial_{d+1}^2$. The propagator reads

$$< f_2|(\frac{m^2-s}{2}-\frac{\partial_{d+1}^2}{2})^{-1}|f_1>=< f_2|\int_0^\infty e^{-\frac{t}{2}(m^2-s-\partial_{d+1}^2)}|f_1> \tag{A1}$$

The evolution operator $e^{t\frac{\partial_{d+1}^2}{2}}$ gives the solution of the heat equation $\frac{\partial}{\partial t}|\psi>=-H|\psi>$ where $H=-\frac{\partial_{d+1}^2}{2}$. This can be written as the path integral in euclidean time t

$$< f_2|(\frac{m^2-s-\partial_{d+1}^2}{2})^{-1}|f_1>=\int_0^\infty e^{-t\frac{(m^2-s)}{2}}\int_{f_1}^{f_2}\mathcal{D}x(t)e^{-\frac{1}{2}\int_0^t\dot{x}^2d\tau} \tag{A2}$$

The path integral corresponds to Brownian trajectories from f_1 to f_2. To evaluate this path integral, we shall use a saddle point approximation. First of all, the path integral is dominated by paths satisfying the equation of motion

$$\ddot{x}=0 \tag{A3}$$

This is the equation of motion of a point particle in euclidean time, i.e. an instanton going from f_1 to f_2. This instanton is therefore

$$x(\tau)=f_1+p\tau \tag{A4}$$

where

$$p=\frac{f_2-f_1}{t} \tag{A5}$$

Substituting in the Green function, one gets

$$< f_2|(m^2-s-\partial_{d+1}^2)^{-1}|f_1>=\frac{1}{2}\int_0^\infty dte^{-t\frac{(m^2-s)}{2}-\frac{1}{2}\frac{|f_2-f_1|^2}{t}}\int_0^0\mathcal{D}x(t)e^{-\frac{1}{2}\int_0^t d\tau\dot{x}^2d\tau} \tag{A6}$$

The path integral concerns closed trajectories returning to the origin in a time t, its value is $(\sqrt{2\pi t})^{-1}$. The t integral can be evaluated using a saddle point approximation. First of all, notice that this integral converges when $s<m^2$, i.e. the energy of the particle is not large enough to create a real particle. In that case the propagator between f_1 and f_2 corresponds to the emission of an instanton. The saddle point equation reads

$$m^2-s-\frac{|f_2-f_2|^2}{t^2}=0 \tag{A7}$$

which implies that

$$t = \frac{R}{\sqrt{m^2 - s}} \tag{A8}$$

Using this result, it is easy to rewrite the saddle point equation

$$s + p^2 = m^2 \tag{A9}$$

This is the mass relation for an instanton of mass m^2 and Euclidian $(d+1)$ momentum p. We can now get the propagator

$$< f_2|(m^2 - s - \partial_{d+1}^2)^{-1}|f_1 >\sim \frac{1}{\sqrt{m^2 - s}} e^{-\sqrt{m^2-s}|f_2 - f_1|} \tag{A10}$$

We see that the saddle point approximation of the Green function gives for the scattering amplitude $T = -\lambda^2 < f_2|(m^2 - s - \partial_{d+1}^2)^{-1}|f_1 >$. Moreover, one can interpret this decay as the propagation of an instanton between the two fixed points.

Let us now analyse the opposite situation when $s > m^2$. In that case the previous integrals are not convergent. One can nevertheless analytically continue the propagator which is now dominated by the exchange of a real particle, i.e. a soliton. The contour of integration can be deformed to the imaginary axis

$$< f_2|(\frac{m^2 - s - \partial_{d+1}^2}{2})^{-1}|f_1 >= \int_0^{i\infty} dt e^{-t\frac{m^2 - s - \partial_{d+1}^2}{2}} \tag{A11}$$

when $s < m^2$ as $-\partial_{d+1}^2$ is a positive operator. Let us now put $t = i\tau$ where τ is a real time parameter. This implies that

$$< f_2|(\frac{m^2 - s - \partial_{d+1}^2}{2})^{-1}|f_1 >= i\int_0^{\infty} d\tau e^{-i\tau\frac{(m^2 - s)}{2}} e^{i\tau\frac{\partial_{d+1}^2}{2}} \tag{A12}$$

The evolution operator $e^{i\tau\frac{\partial_{d+1}^2}{2}}$ corresponds to the Schrodinger equation $i\frac{\partial}{\partial\tau}|\psi >= H|\psi >$ where $H = -\frac{\partial_{d+1}^2}{2}$. One can write the evolution operator as a path integral

$$< f_2|(\frac{m^2 - s - \partial_{d+1}^2}{2})^{-1}|f_1 >= i\int_0^{\infty} d\tau e^{-i\tau\frac{(m^2 - s)}{2}} \int_{f_1}^{f_2} \mathcal{D}x(\tau) e^{\frac{i}{2}\int_0^\tau \dot{x}^2 d\tau_0} \tag{A13}$$

This is the representation of the propagator when $s > m^2$ Notice that now the sum is over classical trajectories of a real particle. As before the path integral is dominated by solitons satisfying

$$\ddot{x} = 0 \tag{A14}$$

whose solution is

$$x = f_1 + p\tau_0 \tag{A15}$$

where $p = \frac{f_2 - f_1}{\tau}$. The propagator becomes

$$< f_2 | (\frac{m^2 - s - \partial^2_{d+1}}{2})^{-1} | f_1 >= i \int_0^\infty d\tau (2\pi i \tau)^{-\frac{1}{2}} e^{-i\tau \frac{(m^2 - s)}{2} + \frac{i}{2} p^2 \tau} \qquad (A16)$$

The saddle point equation becomes

$$m^2 - s + \frac{|f_2 - f_1|^2}{\tau^2} = 0 \qquad (A17)$$

whose solution is given by

$$\tau = \frac{|f_2 - f_1|}{\sqrt{s - m^2}} \qquad (A18)$$

Notice that the saddle point equation reads

$$s - p^2 = m^2 \qquad (A19)$$

the mass relation for a particle of mass m. Hence the propagator corresponds to the emission of a real particle between f_1 and f_2. The integral is obtained after deforming the integration contour. It goes through the saddle point, the angle between the real line and the contour being $-\frac{\pi}{4}$. The propagator becomes

$$< f_2 | (\frac{m^2 - s - \partial^2_{d+1}}{2})^{-1} | f_1 >\sim (s - m^2)^{-\frac{1}{2}} e^{i\sqrt{s - m^2}|f_2 - f_1|} \qquad (A20)$$

The same results appear in the context of orbifolds.

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